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THE EQUATIONS OF SOME DISPERSIONLESS LIMIT

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ABSTRACT. These equations are the generalized equations of several dispersionless equations. A complete table for $p \leq 10$ is provided.

1. INTRODUCTION

It is well-known that a lot of nonlinear solitonic equations can be transformed into certain Hirota type bilinear equations [17]. The τ-function of the KP hierarchy can be characterized by the Hirota equations and the Plücker relations are given from these equations. The differential Fay identity which has quasi-classical limit, is a part of the Plücker relations. The leading term of the quasi-classical limit of the differential Fay identity satisfies an identity [22] and from the identity EQUATION$(\cdot, \infty)$ is extracted.* Therefore, EQUATION$(\cdot, \infty)$ is a subset of the dispersionless KP hierarchy. EQUATION(p, q) are derived from EQUATION(p, ∞). EQUATION$(\cdot, 2)$ can be regarded as a subset of dispersionless KdV hierarchy. We can easily show that EQUATION$(4, 3)$ is a dispersionless Boussinesq equation. Therefore, EQUATION$(\cdot, 3)$ can be regarded as a subset of dispersionless Boussinesq hierarchy. EQUATION$(\cdot, q)$ for $q > 3$ is a whole new set of dispersionless equations which can be regarded as a subset of new hierarchy which may have some useful application.

2. THE FORMULA

Let us use F_{mn} instead of $\dfrac{\partial^2}{\partial t_m \partial t_n}\left(F(t_1, \ldots, t_r, \ldots)\right).$

Definition 2.1.   EQUATION(p, q):

$$\sum_{\substack{0 < i_1 < \cdots < i_{k_p} \\ (i_1+1)n_{i_1} + \cdots + (i_{k_p}+1)n_{i_{k_p}} = p}} \left(\left(\sum_{j=1}^{k_p} n_{i_j} - 1\right)! \prod_{j=1}^{k_p} \frac{(-F_{1i_j})^{n_{i_j}}}{n_{i_j}!} \right)$$

$$+ \sum_{m+n=p} \frac{F_{mn}}{mn} = 0.$$

where the terms having $\frac{\partial}{\partial t_q}, \ldots, \frac{\partial}{\partial t_{kq}}, \ldots$ vanish.

* (Caution)

This notation is used only for technical simplicity of expression since the formula was found by the author recently (Fall, 1994) [5]. This is another way of using equation numbers. Therefore, the author strongly recommends careful use of the notation until the formula is well-known. (Of course, given notation has no meaning elsewhere like other usual equation numbers.)

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3. EQUATION$(4, q)$

One can easily show that EQUATION$(4,\infty)$ is a dispersionless KP and EQUA-TION(4,2) is a dispersionless KdV.

Consider a dispersionless Boussinesq equation

$$(uu_x)_x + \frac{1}{2}u_{yy} = 0. \tag{3.1}$$

If we set $t_1 = x$ and $t_2 = y$, then EQUATION(4,3) is

$$\frac{1}{2}(F_{xx})^2 + \frac{1}{4}F_{yy} = 0. \tag{3.2}$$

Differentiate (3.2) with respect to x. Then we get

$$F_{xx}F_{xxx} + \frac{1}{4}F_{xyy} = 0. \tag{3.3}$$

Setting $u = 2F_{xx}$, (3.3) becomes

$$\left(\frac{u}{2}\frac{u_x}{2}\right)_x + \frac{1}{4}\left(\frac{u_{yy}}{2}\right) = 0.$$

which is the same as (3.1).

4. DISCUSSION

We could get the useful expression of the generalized equations of the dispersionless limit of KdV, KP and Boussinesq equations. And one can get a specific equation for each (p, q). Furthermore, new hierarchies are derived from EQUATION$(\cdot, q)$ for $q > 3$. For further research, a table of equations are provided. By definition, EQUATION(p, q) is the same as EQUATION(p, ∞) for $q \geq p$.

TABLE.   EQUATIONS FOR (p, q).

$(4,\infty)$ $\frac{1}{2}F_{11}{}^2 - \frac{1}{3}F_{13} + \frac{1}{4}F_{22} = 0$

$(5,\infty)$ $F_{11}F_{12} - \frac{1}{2}F_{14} + \frac{1}{3}F_{23} = 0$

$(6,\infty)$ $\frac{1}{3}F_{11}{}^3 - \frac{1}{2}F_{12}{}^2 - F_{11}F_{13} + \frac{3}{5}F_{15} - \frac{1}{9}F_{33} - \frac{1}{4}F_{24} = 0$

$(7,\infty)$ $F_{11}{}^2F_{12} - F_{12}F_{13} - F_{11}F_{14} + \frac{2}{3}F_{16} - \frac{1}{6}F_{34} - \frac{1}{5}F_{25} = 0$

$(8,\infty)$ $\frac{1}{4}F_{11}{}^4 - F_{11}F_{12}{}^2 - F_{11}{}^2F_{13} + \frac{1}{2}F_{13}{}^2 + F_{12}F_{14} + F_{11}F_{15}$

$\qquad\quad - \frac{5}{7}F_{17} + \frac{1}{16}F_{44} + \frac{2}{15}F_{35} + \frac{1}{6}F_{26} = 0$

$(9,\infty)$ $F_{11}{}^3F_{12} - \frac{1}{3}F_{12}{}^3 - 2F_{11}F_{12}F_{13} - F_{11}{}^2F_{14} + F_{13}F_{14}$

$\qquad\quad + F_{12}F_{15} + F_{11}F_{16} - \frac{3}{4}F_{18} + \frac{1}{10}F_{45} + \frac{1}{9}F_{36} + \frac{1}{7}F_{27} = 0$

$(10,\infty)$ $\frac{1}{5}F_{11}{}^5 - \frac{3}{2}F_{11}{}^2F_{12}{}^2 - F_{11}{}^3F_{13} + F_{12}{}^2F_{13} + F_{11}F_{13}{}^2$

$+2F_{11}F_{12}F_{14} - \frac{1}{2}F_{14}{}^2 + F_{11}{}^2F_{15} - F_{13}F_{15} - F_{12}F_{16} - F_{11}F_{17}$

$+\frac{7}{9}F_{19} - \frac{1}{25}F_{55} - \frac{1}{12}F_{46} - \frac{2}{21}F_{37} - \frac{1}{8}F_{28} = 0$

$(4,2)$ $\frac{1}{2}F_{11}{}^2 - \frac{1}{3}F_{13} = 0$

$(5,2)$ $0 = 0$

$(6,2)$ $\frac{1}{3}F_{11}{}^3 - F_{11}F_{13} + \frac{3}{5}F_{15} - \frac{1}{9}F_{33} = 0$

$(7,2)$ $0 = 0$

$(8,2)$ $\frac{1}{4}F_{11}{}^4 - F_{11}{}^2F_{13} + \frac{1}{2}F_{13}{}^2 + F_{11}F_{15} - \frac{5}{7}F_{17} + \frac{2}{15}F_{35} = 0$

$(9,2)$ $0 = 0$

$(10,2)$ $\frac{1}{5}F_{11}{}^5 - F_{11}{}^3F_{13} + F_{11}F_{13}{}^2 + F_{11}{}^2F_{15} - F_{13}F_{15} - F_{11}F_{17}$

$+\frac{7}{9}F_{19} - \frac{1}{25}F_{55} - \frac{2}{21}F_{37} = 0$

$(4,3)$ $\frac{1}{2}F_{11}{}^2 + \frac{1}{4}F_{22} = 0$

$(5,3)$ $F_{11}F_{12} - \frac{1}{2}F_{14} = 0$

$(6,3)$ $\frac{1}{3}F_{11}{}^3 - \frac{1}{2}F_{12}{}^2 + \frac{3}{5}F_{15} - \frac{1}{4}F_{24} = 0$

$(7,3)$ $F_{11}{}^2F_{12} - F_{11}F_{14} - \frac{1}{5}F_{25} = 0$

$(8,3)$ $\frac{1}{4}F_{11}{}^4 - F_{11}F_{12}{}^2 + F_{12}F_{14} + F_{11}F_{15} - \frac{5}{7}F_{17} + \frac{1}{16}F_{44} = 0$

$(9,3)$ $F_{11}{}^3F_{12} - \frac{1}{3}F_{12}{}^3 - F_{11}{}^2F_{14} + F_{12}F_{15} - \frac{3}{4}F_{18} + \frac{1}{10}F_{45} + \frac{1}{7}F_{27} = 0$

$(10,3)$ $\frac{1}{5}F_{11}{}^5 - \frac{3}{2}F_{11}{}^2F_{12}{}^2 + 2F_{11}F_{12}F_{14} - \frac{1}{2}F_{14}{}^2 + F_{11}{}^2F_{15} - F_{11}F_{17}$

$-\frac{1}{25}F_{55} - \frac{1}{8}F_{28} = 0$

$(5,4)$ $F_{11}F_{12} + \frac{1}{3}F_{23} = 0$

$(6,4)$ $\frac{1}{3}F_{11}{}^3 - \frac{1}{2}F_{12}{}^2 - F_{11}F_{13} + \frac{3}{5}F_{15} - \frac{1}{9}F_{33} = 0$

$(7,4)$ $F_{11}{}^2F_{12} - F_{12}F_{13} + \frac{2}{3}F_{16} - \frac{1}{5}F_{25} = 0$

$(8,4)$ $\frac{1}{4}F_{11}{}^4 - F_{11}F_{12}{}^2 - F_{11}{}^2F_{13} + \frac{1}{2}F_{13}{}^2 + F_{11}F_{15} - \frac{5}{7}F_{17} + \frac{2}{15}F_{35} + \frac{1}{6}F_{26} = 0$

$(9,4)$ $F_{11}{}^3F_{12} - \frac{1}{3}F_{12}{}^3 - 2F_{11}F_{12}F_{13} + F_{12}F_{15} + F_{11}F_{16} - +\frac{1}{9}F_{36} + \frac{1}{7}F_{27} = 0$

$(10,4)$ $\frac{1}{5}F_{11}{}^5 - \frac{3}{2}F_{11}{}^2F_{12}{}^2 - F_{11}{}^3F_{13} + F_{12}{}^2F_{13} + F_{11}F_{13}{}^2 + F_{11}{}^2F_{15}$

$$-F_{13}F_{15} - F_{12}F_{16} - F_{11}F_{17} + \tfrac{7}{9}F_{19} - \tfrac{1}{25}F_{55} - \tfrac{2}{21}F_{37} = 0$$

(6,5) $\tfrac{1}{3}F_{11}{}^3 - \tfrac{1}{2}F_{12}{}^2 - F_{11}F_{13} - \tfrac{1}{9}F_{33} - \tfrac{1}{4}F_{24} = 0$

(7,5) $F_{11}{}^2F_{12} - F_{12}F_{13} - F_{11}F_{14} + \tfrac{2}{3}F_{16} - \tfrac{1}{6}F_{34} = 0$

(8,5) $\tfrac{1}{4}F_{11}{}^4 - F_{11}F_{12}{}^2 - F_{11}{}^2F_{13} + \tfrac{1}{2}F_{13}{}^2 + F_{12}F_{14} - \tfrac{5}{7}F_{17} + \tfrac{1}{16}F_{44} + \tfrac{1}{6}F_{26} = 0$

(9,5) $F_{11}{}^3F_{12} - \tfrac{1}{3}F_{12}{}^3 - 2F_{11}F_{12}F_{13} - F_{11}{}^2F_{14} + F_{13}F_{14} + F_{11}F_{16}$
$\qquad - \tfrac{3}{4}F_{18} + \tfrac{1}{9}F_{36} + \tfrac{1}{7}F_{27} = 0$

(10,5) $\tfrac{1}{5}F_{11}{}^5 - \tfrac{3}{2}F_{11}{}^2F_{12}{}^2 - F_{11}{}^3F_{13} + F_{12}{}^2F_{13} + F_{11}F_{13}{}^2 + 2F_{11}F_{12}F_{14}$
$\qquad - \tfrac{1}{2}F_{14}{}^2 - F_{12}F_{16} - F_{11}F_{17} + \tfrac{7}{9}F_{19} - \tfrac{1}{12}F_{46} - \tfrac{2}{21}F_{37} - \tfrac{1}{8}F_{28} = 0$

(7,6) $F_{11}{}^2F_{12} - F_{12}F_{13} - F_{11}F_{14} - \tfrac{1}{6}F_{34} - \tfrac{1}{5}F_{25} = 0$

(8,6) $\tfrac{1}{4}F_{11}{}^4 - F_{11}F_{12}{}^2 - F_{11}{}^2F_{13} + \tfrac{1}{2}F_{13}{}^2 + F_{12}F_{14} + F_{11}F_{15} - \tfrac{5}{7}F_{17} + \tfrac{1}{16}F_{44}$
$\qquad + \tfrac{2}{15}F_{35} = 0$

(9,6) $F_{11}{}^3F_{12} - \tfrac{1}{3}F_{12}{}^3 - 2F_{11}F_{12}F_{13} - F_{11}{}^2F_{14} + F_{13}F_{14} + F_{12}F_{15}$
$\qquad - \tfrac{3}{4}F_{18} + \tfrac{1}{10}F_{45} + \tfrac{1}{7}F_{27} = 0$

(10,6) $\tfrac{1}{5}F_{11}{}^5 - \tfrac{3}{2}F_{11}{}^2F_{12}{}^2 - F_{11}{}^3F_{13} + F_{12}{}^2F_{13} + F_{11}F_{13}{}^2 + 2F_{11}F_{12}F_{14}$
$\qquad - \tfrac{1}{2}F_{14}{}^2 + F_{11}{}^2F_{15} - F_{13}F_{15} - F_{11}F_{17} + \tfrac{7}{9}F_{19} - \tfrac{1}{25}F_{55} - \tfrac{2}{21}F_{37} - \tfrac{1}{8}F_{28} = 0$

(8,7) $\tfrac{1}{4}F_{11}{}^4 - F_{11}F_{12}{}^2 - F_{11}{}^2F_{13} + \tfrac{1}{2}F_{13}{}^2 + F_{12}F_{14} + F_{11}F_{15}$
$\qquad + \tfrac{1}{16}F_{44} + \tfrac{2}{15}F_{35} + \tfrac{1}{6}F_{26} = 0$

(9,7) $F_{11}{}^3F_{12} - \tfrac{1}{3}F_{12}{}^3 - 2F_{11}F_{12}F_{13} - F_{11}{}^2F_{14} + F_{13}F_{14} + F_{12}F_{15}$
$\qquad + F_{11}F_{16} - \tfrac{3}{4}F_{18} + \tfrac{1}{10}F_{45} + \tfrac{1}{9}F_{36} = 0$

(10,7) $\tfrac{1}{5}F_{11}{}^5 - \tfrac{3}{2}F_{11}{}^2F_{12}{}^2 - F_{11}{}^3F_{13} + F_{12}{}^2F_{13} + F_{11}F_{13}{}^2 + 2F_{11}F_{12}F_{14}$
$\qquad - \tfrac{1}{2}F_{14}{}^2 + F_{11}{}^2F_{15} - F_{13}F_{15} - F_{12}F_{16} + \tfrac{7}{9}F_{19} - \tfrac{1}{25}F_{55} - \tfrac{1}{12}F_{46} - \tfrac{1}{8}F_{28} = 0$

(9,8) $F_{11}{}^3F_{12} - \tfrac{1}{3}F_{12}{}^3 - 2F_{11}F_{12}F_{13} - F_{11}{}^2F_{14} + F_{13}F_{14} + F_{12}F_{15}$
$\qquad + F_{11}F_{16} + \tfrac{1}{10}F_{45} + \tfrac{1}{9}F_{36} + \tfrac{1}{7}F_{27} = 0$

(10,8) $\tfrac{1}{5}F_{11}{}^5 - \tfrac{3}{2}F_{11}{}^2F_{12}{}^2 - F_{11}{}^3F_{13} + F_{12}{}^2F_{13} + F_{11}F_{13}{}^2 + 2F_{11}F_{12}F_{14}$

$$-\tfrac{1}{2}F_{14}{}^2 + F_{11}{}^2F_{15} - F_{13}F_{15} - F_{12}F_{16} - F_{11}F_{17} + \tfrac{7}{9}F_{19} - \tfrac{1}{25}F_{55}$$

$$-\tfrac{1}{12}F_{46} - \tfrac{2}{21}F_{37} = 0$$

(10,9) $\qquad \tfrac{1}{5}F_{11}{}^5 - \tfrac{3}{2}F_{11}{}^2F_{12}{}^2 - F_{11}{}^3F_{13} + F_{12}{}^2F_{13} + F_{11}F_{13}{}^2 + 2F_{11}F_{12}F_{14}$

$$-\tfrac{1}{2}F_{14}{}^2 + F_{11}{}^2F_{15} - F_{13}F_{15} - F_{12}F_{16} - F_{11}F_{17} - \tfrac{1}{25}F_{55} - \tfrac{1}{12}F_{46}$$

$$-\tfrac{2}{21}F_{37} - \tfrac{1}{8}F_{28} = 0$$

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